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On trigonal non-Gorenstein curves with zero Maroni invariant

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Abstract

By means of canonical models we answer some questions on trigonal non-Gorenstein curves with zero Maroni invariant: the number of non-Gorenstein points, the kind of such singularities, possible canonical models, uniqueness and number of base points of a g_3^1 on the curve.

Keywords: Trigonal curve; Non-Gorenstein curve; Canonical model; Base point

Introduction

In the early 1950s, M. Rosenlicht published the article “Equivalence Relations on Algebraic Curves” [10], the first of a trilogy that, together with Lang’s works, motivated the famous book *Groupes Algébriques et Corps des Classes* [13] of J.-P. Serre. On page 535 of the mentioned paper is defined a certain curve C'—which we call here, as in [2], (*Rosenlicht’s*) *canonical model*—that will play a central role throughout these lines.

More precisely, given an algebraic curve C of arithmetic genus g, its canonical linear system induces a morphism $\widetilde{C} \to \mathbb{P}^{g-1}$ where $\widetilde{C}$ is the nonsingular model. The image of this morphism is the above C'. When a nonhyperelliptic curve C is Gorenstein, C and C' are isomorphic. Otherwise, C' is just birationally equivalent to C, but can give important data about the latter. What we do here is exactly this: to study trigonal non-Gorenstein curves by the analysis of their canonical models.

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At first glance, we already have a problem: what is *trigonal* in this context? E. Ballico summarizes in [3, Definition 3.7] many—we counted twelve—definitions of gonality that we find in the literature which have no reason to agree on non-Gorenstein case. We follow the same language used in [12,14,15], whose results we want to generalize for non-Gorenstein curves. There are already non-Gorenstein examples in [12, cf. Examples 4.1–4.3] and we use the term *gonality* in the same way.

So a *divisor* will be a coherent fractional ideal sheaf, or equivalently (cf. [7, Corollary 8.4.3]), a torsion free sheaf of rank 1. Moreover, we regard as *effectives* the divisors that contain the structure sheaf, and not, as usual, the converse, which is an irrelevant fact for nonsingular curves, but not in the singular case (cf. [16, Introduction]).

Such an approach was proven to be very successful in [12] when dealing with gonality within the same spirit of Maroni's *"come è noto:" "A nonhyperelliptic curve is trigonal if it admits a pencil of degree* 3*"* (cf. [9, p. 333]).

Now we describe the contents of this paper. The first section is devoted to Rosenlicht's canonical model. The main result is Theorem 1.4 where we show that the canonical model is isomorphic to the canonical blowup (not the common one) and, as a consequence, when the curve is non-Gorenstein, the difference between its genus and the one of the canonical model is at least 2. Though a simple consequence of Rosenlicht's theorem, we point out that its proof follows from tools we now have available.

In the following section we study curves that admit a base point g^1_2, which we call *basic hyperelliptic*. A characterization of these curves is easily read off from Clifford's theorem for singular curves (cf. [8, Appendix]). We give another (geometric) one using [12] and obtain results which perfectly match with hyperelliptic (Gorenstein) curves in Theorem 2.4. On the other hand, the g^1_2 is not cut out by the rule of the surface where the curve lies (a cone) but only its support. Despite its intrinsic interest, we directed the section to the next one, since basic hyperelliptic curves are possible canonical models for trigonal non-Gorenstein curves.

We leave for the last section the main subject of this paper, that is, the trigonal case. We begin by separating curves of genus 3 and we have a reason to do so: it does not make much sense to speak of Maroni invariant for a g^1_3 of such curves (it would be zero) when they are trigonal since they trivially satisfy the theorem which defines the invariant. Besides, the canonical models of these curves are plane, as opposed to curves of higher genus, where the canonical model lives on a scroll, possibly degenerated, i.e., a cone. There are few possibilities that can be entirely covered, leading us to the following result: such curves have gonality at most 3, and if it is exactly 3, their canonical models are elliptic (Theorem 3.1).

Afterwards, we answer some questions which we summarize here within a single statement:

Theorem. *Every trigonal non-Gorenstein curve carrying a zero Maroni g^1_3 is almost Gorenstein with only one non-Gorenstein point, having a* (*possibly basic*) *hyperelliptic curve as canonical model. If the genus of the canonical model is greater or equal to* 2*, then the curve admits a unique Maroni zero g^1_3, having also a unique base point, which is the non-Gorenstein point. If the canonical model is basic hyperelliptic, then there are no other g^1_3's besides the unique one with zero Maroni invariant.*

The terminology *almost Gorenstein* and even *Kunz* are used in the same manner as [6] and others, and is what we refer here as *kind* of singularity (since *type* is forbidden). We also give two characterization results: one of them on trigonal Kunz curves (Theorem 3.7), and the other on unibranch ones (Theorem 3.8).

We always give examples using monomial curves. They are very particular and therefore dangerous, but may be the fastest way of getting the intuition needed. And we recommend [1] as an example that we don't need to go further to get strong results.

1. Rosenlicht's canonical model

1.1. Linear systems

By a *curve*, to simplify, we will always mean an integral one-dimensional scheme of finite type and complete over an algebraically closed field.

So let C be a curve defined over k. A *divisor* in C is a coherent (fractional) ideal sheaf $\mathfrak{a}$ identified with the formal product of its stalks

$$\mathfrak{a} = \prod_{P \in C} \mathfrak{a}_P,$$

and $\mathrm{Div}(C)$ will be the set of divisors of the curve, where we can define both product and division operations stalk by stalk. The structure sheaf $\mathcal{O} := \mathcal{O}_C$ is the neutral element of the product, and also right-neutral of the division.

As the same manner, we establish in $\mathrm{Div}(C)$ a partial order, i.e., $\mathfrak{a} \leqslant \mathfrak{b}$ whenever $\mathfrak{a}_P \subset \mathfrak{b}_P$ for each $P \in C$, and we define the *degree* of a divisor as

$$\deg \mathfrak{a} = \chi(\mathfrak{a}) - \chi(\mathcal{O}),$$

where χ is the Euler characteristic function. In particular, if $\mathfrak{a} \geqslant \mathfrak{b}$ then $\deg \mathfrak{a} - \deg \mathfrak{b} = \sum_{P \in C} \dim_k \mathfrak{a}_P / \mathfrak{b}_P$.

A divisor is called *locally principal*, or *Cartier*, if its stalks are principal ideals of the respective local rings. The term *invertible* is also used since only these divisors are invertible with respect to the product. We omit the word "locally" if the generators of each stalk are always the same and for a given rational function $x \in k(C)^*$ we associate the principal divisor

$$\mathrm{div}(x) := \prod_{P \in C} x^{-1} \mathcal{O}_P.$$

Therefore, the space of global sections $H^0(C, \mathfrak{a}) := \bigcap_{P \in C} \mathfrak{a}_P$ can be seen as

$$H^0(\mathfrak{a}, C) = \left\{ x \in k(C)^* \;\middle|\; \mathrm{div}(x) \cdot \mathfrak{a} \geqslant \mathcal{O} \right\} \cup \{0\}$$

and we denote $h^0(C,\mathfrak{a}) := \dim H^0(C,\mathfrak{a})$ and $|\mathfrak{a}|$ to be the class of positive divisors linearly equivalent to $\mathfrak{a}$, that is,

$$|\mathfrak{a}| := \left\{ \mathrm{div}(x)\cdot\mathfrak{a} \;\middle|\; x \in H^0(C,\mathfrak{a}) \setminus 0 \right\}.$$

We can associate to each finite dimensional k-vector space $V = \langle x_0,\ldots,x_n\rangle \subset k(C)$ of dimension $n+1 \geqslant 2$ the divisor

$$\mathcal{O}(V) = \mathcal{O}(x_0,\ldots,x_n) := \prod_{P\in C} x_0\mathcal{O}_P + \cdots + x_n\mathcal{O}_P, \tag{1}$$

which plays the same role of $\mathrm{div}(x)$ when we are given two or more rational functions.

We will refer to a *linear system* in a curve C as being a linear subspace of $|\mathfrak{a}|$, i.e., a set of the form

$$\mathcal{L} = \mathcal{L}(\mathfrak{a}, V) := \left\{ \mathrm{div}(x)\cdot\mathfrak{a} \;\middle|\; x \in V \setminus 0 \right\},$$

where V is a vector subspace of $H^0(C,\mathfrak{a})$. In the case where $V = H^0(C,\mathfrak{a})$ we say that the system is *complete*.

To each linear system is associated a *degree* and a *dimension* defined respectively by $d := \deg\mathfrak{a}$ and $n := \dim V - 1$ and, in this sense, the notation g_d^n stands for the expression: "linear system of degree d and dimension n."

A point $P \in C$ is called a *base point* of the linear system if it is in the support of every divisor of $\mathcal{L}$, i.e., $\mathfrak{b}_P \supsetneqq \mathcal{O}_P$ for each $\mathfrak{b} \in \mathcal{L}$. Now given a linear system $\mathcal{L} = \mathcal{L}(\mathfrak{a}, V)$, consider the divisor $\mathcal{O}(V)$ as defined in (1). Since $V \subset H^0(C, \mathcal{O}(V))$, we can define the *spanned linear system associated to* $\mathcal{L}$ as

$$\mathcal{L}_{\mathrm{sp.}} := \left\{ \mathrm{div}(x)\cdot\mathcal{O}(V) \;\middle|\; x \in V \setminus 0 \right\}.$$

When $\mathcal{L} = \mathcal{L}_{\mathrm{sp.}}$ we say that $\mathcal{L}$ is *spanned*. Otherwise, though having the same dimension, $\mathcal{L}_{\mathrm{sp.}}$ has degree strictly smaller than $\mathcal{L}$. A base point of $\mathcal{L}$ which is not base point of $\mathcal{L}_{\mathrm{sp.}}$ is said to be a *removable* base point of $\mathcal{L}$. The reader can easily check that non-removable base points are always singular. A linear system will be defined as *base point free* if it does not have them.

Finally, we will define the *gonality* of a curve as the smallest d such that the curve admits a g_d^1. We say that such linear system *computes* the gonality of the curve and, if so, it is obviously spanned and, if $d \leqslant 3$, is also complete, as can be easily seen from Clifford's theorem for instance.

1.2. The canonical model

Let $\widetilde{C}$ be the nonsingular model of C and $\pi : \widetilde{C} \to C$ the natural projection. Denoting $\widetilde{\mathcal{O}} := \pi_*(\mathcal{O}_{\widetilde{C}})$, we have

$$\widetilde{\mathcal{O}}_P = \bigcap_{\widetilde{P}|P} \mathcal{O}_{\widetilde{P}},$$

which coincides with the integral closure of $\mathcal{O}_P$ in $k(C)$. By direct image we can consider $\mathrm{Div}(\widetilde{C}) \subset \mathrm{Div}(C)$ and such divisors are also called $\widetilde{\mathcal{O}}$-divisors. The greatest $\widetilde{\mathcal{O}}$-divisor lesser than $\mathcal{O}$ is the *conductor* $\mathfrak{c}$ whose stalks correspond to the local conductors $\mathfrak{c}_P = (\mathcal{O}_P : \widetilde{\mathcal{O}}_P)$.

For each nonzero meromorphic differential $\lambda \in \Omega_{C|k}$ we define the divisor $\mathrm{div}(\lambda) := \omega$ where each stalk is the greatest among the $\mathcal{O}_P$-ideals ω_P's in $k(C)$ such that $\sum_{\widetilde{P}|P} \mathrm{Res}_{\widetilde{P}}(x \cdot \lambda) = 0$ for every $x \in \omega_P$. A differential divisor ω is named a *canonical divisor* and corresponds to a dualizing sheaf, satisfying the *local duality*: for each $\mathcal{O}_P$-ideals $\mathfrak{a}_P \supset \mathfrak{b}_P$,

$$(\omega_P : \mathfrak{b}_P)/(\omega_P : \mathfrak{a}_P) \cong \mathrm{Hom}_k\big((\mathfrak{a}_P/\mathfrak{b}_P), k\big),$$

in particular, $h^0(C, (\omega : \mathfrak{a})) = h^1(C, \mathfrak{a}) := \dim H^1(C, \mathfrak{a})$. The canonical divisors also satisfy the *reciprocity*: for every divisor $\mathfrak{a}$ of C,

$$\omega : (\omega : \mathfrak{a}) = \mathfrak{a}$$

or, in other words, stalks of canonical divisors are canonical ideals of their respective local rings.

For each $P \in C$ there always exists a canonical divisor ω of the curve such that $\mathcal{O}_P \subset \omega_P \subset \widetilde{\mathcal{O}}_P$ and we will say that a canonical divisor like this is *P-normalized*.

Now, we are ready to begin with our specific subject. We start introducing *Gorenstein* curves. For this we note that *genus* will always mean arithmetic genus and we use in general the notation $\phi_{\mathfrak{a}}$ for the morphism—when it can be defined—naturally associated to $|\mathfrak{a}|$.

Definition 1.1. A curve C of genus g is *Gorenstein* if it satisfies the well-known equivalent conditions:

(i) $\dim \widetilde{\mathcal{O}}_P/\mathcal{O}_P = \dim \mathcal{O}_P/\mathfrak{c}_P$ for each $P \in C$.
(ii) For each $P \in C$ we have $\omega_P = \mathcal{O}_P$ for every P-normalized canonical divisor ω of C.
(iii) The canonical divisors of C are invertibles.
(iv) $g = 0$ or there exists a morphism $\phi_\omega : C \to \mathbb{P}^{g-1}$.

We will refer to curves that are not Gorenstein simply as *non-Gorenstein*, the same holding for points of curves that do not satisfy the requirements (i) and (ii) from the above definition. For every point of any curve it is true that $\dim \widetilde{\mathcal{O}}_P/\omega_P = \dim \mathcal{O}_P/\mathfrak{c}_P$ whenever ω is a P-normalized canonical divisor, as a consequence of local duality. Nevertheless, if the point is non-Gorenstein there exists an *abyss* between $\mathcal{O}_P$ and ω_P that we will denote by

$$\eta_P := \dim \omega_P/\mathcal{O}_P$$

and by η the sum of the η_P's along the points of the curve.

Now we can introduce the *canonical model* of a curve. For this, we will say that a curve is *hyperelliptic* if it admits a base point free g^1_2. In particular, arithmetically rational ($g = 0$) and elliptic ($g = 1$) curves will be considered, for simplicity, as hyperelliptic.

Definition 1.2. Let C be a nonhyperelliptic curve of genus g and ω a canonical divisor of C. The curve $C' := \phi_\omega(\widetilde{C})$ contained in $\mathbb{P}^{g-1}$ is called the *Rosenlicht's canonical model* of the curve C or simply the *canonical model*. Such a curve is unique up to projective equivalence in $\mathbb{P}^{g-1}$.

Also used are the terms *partial normalization* or *dessingularization* for the canonical model, this use being justified by Rosenlicht's theorem enunciated in the sequence, the proof of which can be found in [10, Theorem 17].

Theorem 1.3 (Rosenlicht). *In the conditions of the preceding definition, there exists a morphism $\psi : C' \to C$ such that $\psi \circ \phi_\omega = \pi$.*

From an intrinsic point of view, the above result tells us that for each $P \in C$ the semilocal ring

$$\mathcal{O}'_P := \bigcap_{\widetilde{P}|P} \mathcal{O}_{\phi_\omega(\widetilde{P})}$$

satisfies $\mathcal{O}_P \subset \mathcal{O}'_P \subset \widetilde{\mathcal{O}}_P$.

Now let us consider, given a fixed $P \in C$ and a canonical divisor ω of C, the semilocal ring

$$\bar{\mathcal{O}}_P := \bigcup_{n \geqslant 0} \left({\omega_P}^n : {\omega_P}^n \right),$$

which is known as the blowup of the $\mathcal{O}_P$-ideal ω_P. Since $\mathcal{O}_P \subset \bar{\mathcal{O}}_P \subset \widetilde{\mathcal{O}}_P$, we can define a curve $\bar{C}$ intrinsically by the semilocal rings $\bar{\mathcal{O}}_P$ which we call the *canonical blowup* of C.

If we take a positive P-normalized canonical divisor ω, then $H^0(C, \omega) = \langle x_1, \ldots, x_{g-1}, 1\rangle$ and the x_i's are local coordinate functions at each $\phi_\omega(\widetilde{P})$ such that $\widetilde{P}|P$. So we have

$$\bar{\mathcal{O}}_P = \mathcal{O}_P[x_1, \ldots, x_{g-1}].$$

In fact, as ω is spanned by global sections, $\mathcal{O}_P[x_1, \ldots, x_{g-1}] = \bigcup_{n \geqslant 0} {\omega_P}^n$ which implies the existence of n_0 sufficiently large such that ${\omega_P}^n = {\omega_P}^{n_0}$ for $n \geqslant n_0$, that is, every power of ω_P is in $({\omega_P}^{n_0} : {\omega_P}^{n_0})$. Since ω is P-normalized, we have $({\omega_P}^n : {\omega_P}^n) \subset {\omega_P}^n$ whenever $n \geqslant 0$ and the equality follows.

Theorem 1.4. *For a nonhyperelliptic curve C of genus g we have*:

(i) $C' \cong \bar{C}$.
(ii) *If $P \in C$ is non-Gorenstein, then*

$$\delta_P - \sum_{P'|P} \delta_{P'} > \eta_P,$$

where δ means singularity degree and the P''s are the points of C' over P. In particular, if C is non-Gorenstein, then $g - g' \geqslant 2$ where g' is the genus of C'.

Proof. (i) For each $P \in C$, we will show that $\mathcal{O}'_P = \mathcal{O}_P[x_1, \ldots, x_{g-1}]$. The inclusion "$\supset$" is a consequence of the preceding theorem, since $\mathcal{O}'_P$ is a ring in $k(C)$ which contains $\mathcal{O}_P$ and the x_i's. For "$\subset$" let $\mathcal{O}_{P_1}$ be one of the elements of the local decomposition of the semilocal ring $\mathcal{O}_P[x_i]_{i=1}^{g-1}$ lying under the points $\widetilde{P}_1, \ldots, \widetilde{P}_l$ of $\widetilde{C}$. We have $k[x_1, \ldots, x_{g-1}] \subset \mathcal{O}_P[x_1, \ldots, x_{g-1}] \subset \mathcal{O}_{P_1}$ and hence, for each $j \in \{1, \ldots, l\}$, if $c_{ij} := x_i(\widetilde{P}_j)$ and $\mathfrak{m}$ always meaning the maximal ideal, we have

$$\mathfrak{m}_{\phi_\omega(\widetilde{P}_j)} = (x_1 - c_{1j}, \ldots, x_{g-1} - c_{g-1,j}) \subset \mathfrak{m}_{\widetilde{P}_j} \cap k[x_i, \ldots, x_{g-1}] \subset \mathfrak{m}_{\widetilde{P}_j} \cap \mathcal{O}_{P_1} = \mathfrak{m}_{P_1}$$

and therefore $\mathfrak{m}_{\phi_\omega(\widetilde{P}_j)} = k[x_1, \ldots, x_{g-1}] \cap \mathfrak{m}_{P_1}$, thus $\mathcal{O}_{\phi_\omega(\widetilde{P}_j)} \subset \mathcal{O}_{P_1}$. Hence $\bigcap_{j=1}^{l} \mathcal{O}_{\phi_\omega(\widetilde{P}_j)}$ is contained in $\mathcal{O}_{P_1}$. From the generality of P_1 follows the inclusion.

(ii) Since $\delta_P - \sum_{P'|P} \delta_{P'} = \dim \mathcal{O}'_P/\mathcal{O}_P$, it follows that the equality holds if and only if $\mathcal{O}'_P = \omega_P$ and hence ω_P is a ring, in particular, $\omega_P^2 = \omega_P$ and thus

$$\mathcal{O}_P = (\omega_P : \omega_P) = \left(\omega_P : \omega_P^2\right) = \left((\omega_P : \omega_P) : \omega_P\right) = (\mathcal{O}_P : \omega_P)$$

which implies $\omega_P \subset \mathcal{O}_P$ and P is Gorenstein. $\square$

As a consequence of the above result, $\mathcal{O}'_P$, contrary to what its definition may suggest, only depends on P—for ω is defined stalk by stalk from a differential, which only depends on the nonsingular model, which only depends on P—and is the smallest ring in $k(C)$ that contains ω_P. Therefore, to obtain the canonical model, we can work separately with each singular point, which is a very helpful property and will be used from now on without explicit mention.

Although we have given a "theorem status" to the result, it is of course an immediate consequence of Theorem 1.3. But we note how difficult would be to verify that $\omega_P \subset \mathcal{O}_P[x_1, \ldots, x_{g-1}]$ (which implicitly appears before the theorem's statement) without the property that canonical divisors are spanned by global sections, fact that was not used in [10].

We can also say that the canonical model, viewed as blowup, as opposed to the common one, improves the singularity if and only if it is non-Gorenstein, and here is one more reason to adopt our terminology. In fact, since C' is either Gorenstein or, otherwise, nonhyperelliptic, we can continue the process and construct a sequence

$$C \leftarrow C' \leftarrow C'' \leftarrow \cdots \leftarrow \widetilde{C}$$

that does not necessarily end at $\widetilde{C}$ (we stop when the curve is Gorenstein!). There are plenty of examples that C' need not be Gorenstein in general, from which we choose the following: consider the rational parametrized projective—though denoted as affine—curve $C = (x^7, x^{11}, x^{13}, x^{14}) \subset \mathbb{P}^4$ which is non-Gorenstein at the origin, its unique singular point (the "irrelevant" component x^{14} avoids singularity at the infinity). By some semigroup techniques (cf. [4, Proposition 2.14]), we have $C' \cong (x^7, x^{11}, x^{13}, x^{14}, x^{15})$ and $C'' \cong (x^4, x^6, x^7)$, the latter being Gorenstein. We selected this example because it shows that the abyss η_P, which is a "gorensteiness" parameter, need not decrease at each step of the process, although the curve is "closer" to being Gorenstein. In fact, we have, at the origin, abysses 1 and 4 for, respectively, the curves C and C'.

2. Hyperelliptic non-Gorenstein curves

2.1. Curves on cone

For later use, we recall here a few results about curves on a cone. For more details, see [12].

Let S be the cone on the N-dimensional projective space given by equations

$$S := \left\{ (X_0 : \ldots : X_{N-1} : Y) \in \mathbb{P}^N \;\middle|\; \operatorname{rank}\begin{pmatrix} X_0 & \ldots & X_{N-2} \\ X_1 & \ldots & X_{N-1} \end{pmatrix} < 2 \right\}$$

of vertex $P := (0 : \ldots : 0 : 1)$, which is the union of projective lines

$$L_x := \left\{ \left(1 : x : \ldots : x^{N-1} : y\right) \;\middle|\; y \in k \right\} \cup \{P\}$$

for each $x \in k$ and the line relative to infinity

$$L_\infty := \left\{ (0 : \ldots : 0 : 1 : y) \;\middle|\; y \in k \right\} \cup \{P\}.$$

Consider the local chart $k^2 \cong S \setminus L_\infty$ given, as above, by

$$(x, y) \leftrightarrow \left(1 : x : \ldots : x^{N-1} : y\right).$$

It establishes a 1-1 relation between irreducible projective curves on S distinct from L_∞ and irreducible affine plane curves, that is, to every curve $C \subset S \subset \mathbb{P}^N$ is associated an irreducible equation

$$c_l(x)y^l + \cdots + c_1(x)y + c_0(x) = 0, \quad c_l(x) \neq 0,$$

and conversely.

If we take d to be the smallest integer satisfying

$$\deg c_i(x) \leqslant d - i(N-1), \quad i \in \{0, 1, \ldots, l\}, \tag{2}$$

it can be proven that d coincides with the degree of the curve.

The mentioned reference also shows a way (cf. Formula 3.1) to compute the genus of the curve in terms of its degree and the ambient space dimension N:

$$p_a(C) = (q-1)\left(\deg(C) - 1\right) - \frac{1}{2}q(q-1)(N-1), \tag{3}$$

where q is the smallest integer greater or equal to $\deg(C)/N - 1$.

2.2. Basic hyperelliptic curves

As previously defined, a curve is hyperelliptic if it is equipped with a base point free g_2^1. Removing this hypothesis brings us to the subject of this section, that is, curves carrying a base point g_2^1.

The unique singular curves of genus 1, that is, the plane rational curves of the form $(f(x), xf(x))$ where f is a polynomial of degree 2, are also the unique curves that admit both a base point free and a base point g_2^1, respectively, $|\mathcal{O}(1, f(x))|$ and $|\mathcal{O}(1, x)|$. That they are indeed the only ones, we can see from the result below.

Theorem 2.1. *For a curve of genus greater or equal to* 2, *the following are equivalent*:

(i) *C admits a base point g_2^1.*
(ii) *C is rational with only one singular point P, which also satisfies $\mathfrak{m}_P = \mathfrak{c}_P$.*
(iii) *C is isomorphic to a curve of degree $2g+1$ lying on a cone $S \subset \mathbb{P}^{g+1}$ with a singularity at the vertex.*

Proof. (i) $\Leftrightarrow$ (ii). Follows from Clifford's theorem for singular curves (cf. [8, Appendix]).

(ii) $\Rightarrow$ (iii). Cf. [8, Remark, p. 533].

(iii) $\Rightarrow$ (ii). Set $\deg(C) = 2g+1$ and $N = g+1$ in (3). Since $q = 3$, it follows that g is the genus of the curve. Moreover, using (2), we have that the plane equation of the curve on the cone is given by

$$f(x, y) = c_2(x)y^2 + c_1(x)y + c_0(x) = 0$$

with $\deg c_2(x) \leqslant 1$, $\deg c_1(x) \leqslant g+1$, and $\deg c_0(x) \leqslant 2g+1$.

Since the coordinate functions at the vertex P of the cone are $(1/y, x/y, \ldots, x^g/y)$ and assuming that L_∞ is tangent to the curve at the vertex, if $c_2(x) \neq 0$, we can take it constant and rapidly see that the vertex is nonsingular, and hence $c_2(x) = 0$. On the other hand, supposing that L_∞ is not tangent to the curve at the vertex, we can take $\deg c_1(x) = g+1$ and its roots are the points of $\mathbb{P}^1$ over P. Then we have

$$\mathcal{O}_P = k + \sum_{i=0}^{g} k\frac{x^i}{y} + \mathfrak{m}_P{}^2$$

and thus $\dim \widetilde{\mathcal{O}}_P/\mathfrak{m}_P\widetilde{\mathcal{O}}_P = \deg c_1(x) = g+1$ which implies $\mathfrak{m}_P = \mathfrak{m}_P\widetilde{\mathcal{O}}_P$, that is, the maximal ideal is the conductor of the local ring, and also that the vertex is the unique singular point of C. $\square$

Definition 2.2. A curve satisfying the equivalent conditions of the above result will be called *basic hyperelliptic*.

The basic hyperelliptic curves are then a particular case (rational) of the *singular curves defined by a module* (a positive divisor of $\widetilde{C}$) widely studied by J.-P. Serre in [13], but from a different point of view.

Among the parameterized rational curves $(f(x), xf(x), \ldots, x^g f(x)) \subset \mathbb{P}^{g+1}$, where f is a polynomial of degree $g+1$, we find, up to isomorphism, all basic hyperelliptic curves of genus g. We can also prove—with a little additional effort and using the same normalizing and free coefficients counting method of [11, Theorem 3.1]—that the moduli dimension of the basic hyperelliptic curves of genus g whose singular point has a branch of multiplicity n is $2g-n$.

Corollary 2.3. *Every basic hyperelliptic curve is non-Gorenstein, has $\mathbb{P}^1$ for canonical model, and admits a unique g_2^1.*

Proof. The first statement follows immediately from (ii) of the preceding theorem. For the second, at the singular point P we have $\dim \widetilde{\mathcal{O}}_P/\omega_P$ (which we know is equal to $\dim \mathcal{O}_P/\mathfrak{c}_P$) being 1. This implies that $C' \cong \mathbb{P}^1$ for $\mathcal{O}'_P$ cannot coincide with ω_P when the point is non-Gorenstein (cf. Theorem 1.4(ii)).

And for the remaining one, without loss in generality, let $r_1, \ldots, r_s \in k$ be the points of $\mathbb{P}^1$ over P and let x be the rational function which is identity at finite distance. If we set $\mathfrak{a} := \mathcal{O}_C(1, x)$ we have that $\mathcal{L} := |\mathfrak{a}|$ is a base point g_2^1 for the basic hyperelliptic curve C. Since a g_2^1 of a curve of gonality 2 is complete and spanned, let $\mathcal{L}_0 = |\mathfrak{b}|$ with $\mathfrak{b} = \mathcal{O}_C(1, h(x))$ be another base point g_2^1 on C. We can write $h(x) = \prod (x-r_i)^{n_i} . \prod (x-a_j)^{m_j}$. Since the unique singular point P is necessarily the base point of the linear system, it follows that either $\deg_P \mathfrak{b} = 2$ and then $h(x)$ is of the form $(x-c)/(x-r_{i_0})$ for some c equal to some a_j or r_i provided that $i \neq i_0$; or, if $\deg_P \mathfrak{b} = 1$, then $h(x)$ equals $(x-c)$ or $(x-c)/(x-a_{j_0})$ in case the infinity is or not in the support of $\mathfrak{b}$, what implies the equalities $\mathfrak{b} = \operatorname{div}(x-r_{i_0}) \cdot \mathfrak{a}$ and respectively $\mathfrak{b} = \mathfrak{a}$ or $\mathfrak{b} = \operatorname{div}(x-a_{j_0}) \cdot \mathfrak{a}$; thus $\mathcal{L}_0 = \mathcal{L}$. Hence the g_2^1 is unique since the curve is non-Gorenstein and cannot admit a base point free g_2^1 for it is not hyperelliptic. □

From Theorem 2.1 we see that curves of genus greater or equal to 2 equipped with a base point g_2^1 can lie on a cone of $\mathbb{P}^{g+1}$ with degree $2g+1$ and passing through the vertex. If we want the converse, it is necessary to consider them in $\mathbb{P}^{g+2}$ with degree $2g+2$, i.e., to take a parametrization of the form $C \cong (f(x), xf(x), \ldots, x^g f(x), x^{g+1} f(x))$ and, mutatis mutandi, repeat the proof of the theorem. If we gather the obtained result with [15, Theorem 2.1] we have the following theorem.

Theorem 2.4. *A curve of genus greater or equal to 2 carries a g_2^1 if and only if it is isomorphic to a curve of degree $2g+2$ on the cone $S \subset \mathbb{P}^{g+2}$. The g_2^1 is unique and admits a base point if and only if the* (*isomorphic*) *curve passes through the vertex.*

The missing statement on the above result is that the g_2^1 is cut out by the intersection of C with the lines who rule the cone. But this is not true if the g_2^1 is base point.

In fact, each divisor $\operatorname{div}(x-a) \cdot \mathcal{O}(1, x)$ of the g_2^1 of C is naturally associated to the line L_a of the cone (and $\mathcal{O}(1, x)$ to L_∞), but we cannot arbitrarily choose the degree at each point as our taste. The way to do it formally is the following: for each linear subspace L and each curve C of a projective space, we define the divisor $C.L$ to be the infimum of intersection divisors of the curve with hyperplanes containing L. We wish $C.L_\infty$ and $C.L_a$ have degree 1 at the vertex (or 2 if a is a root of $c_1(x)$), which does not happen: if you compute, for instance, $C.L_\infty$ at the vertex P of S you will see that $(C.L_\infty)_P = \widetilde{\mathcal{O}}_P$ and hence $\deg_P C.L_\infty = \delta_P = g$ (!!). This suggests a definition of intersection divisors that discards the "excess" at non-Gorenstein points.

3. Trigonal non-Gorenstein curves

3.1. Curves of genus 3

A curve will be called *trigonal* if it has gonality 3. In particular, from Theorem 2.1, such curves, if non-Gorenstein, have genus greater or equal to 3.

So let C be a non-Gorenstein curve of genus 3. It has a unique non-Gorenstein point P which, from local duality and Theorem 1.4, must satisfy one of the four cases below, where ω is P-normalized:

	case 1	case 2	case 3	case 4
$\dim \widetilde{\mathcal{O}}_P/\mathcal{O}'_P$	0	0	0	1
$\dim \mathcal{O}'_P/\omega_P$	1	1	2	1
$\dim \omega_P/\mathcal{O}_P$	1	2	1	1
$\dim \mathcal{O}_P/\mathfrak{c}_P$	1	1	2	2

The curve is basic hyperelliptic if and only if it satisfies case 2 and we will see later that if $\dim \omega_P/\mathcal{O}_P = 1$ the same holds for $\dim \mathcal{O}'_P/\omega_P$, avoiding case 3. Only remains the first case—that can correspond either to a curve defined by a module of a nonsingular elliptic curve or to a rational curve with two singularities, one of them being Gorenstein—or the last one: a rational curve whose canonical model is elliptic with a singularity that is over the non-Gorenstein point. So we have proved the second statement of the theorem below.

Theorem 3.1. *Every non-Gorenstein and non basic hyperelliptic curve of genus* 3 *is trigonal with a* (*possibly singular*) *elliptic curve as canonical model.*

Proof. If C satisfies case 1, consider $Q \in C'$ which is not over the unique non-Gorenstein point P of C and $x \in k(C) = k(C')$ such that $\mathcal{O}_{C'}(1, x)$ is a divisor of C' with degree 2 and totally supported at Q (remember that C' is elliptic). Let us consider the divisor $\mathfrak{a} = \mathcal{O}_C(1, x)$ of C. Since $\mathfrak{m}_P = \mathfrak{c}_P$, it follows that $\mathcal{O}_P = k \oplus \mathfrak{c}_P$. But $x \in \widetilde{\mathcal{O}}_P$ and hence $x\mathfrak{c}_P \subset \mathfrak{c}_P$ and $x \notin \mathfrak{c}_P$ because the sum of the conductor orders must be at least 3 for P to be non-Gorenstein and we have $\mathfrak{a}_P = k \oplus kx \oplus \mathfrak{c}_P$ implying $\deg_P \mathfrak{a}_P = 1$ and it follows that $\deg \mathfrak{a} = 3$ and C is trigonal.

For case 4, we will consider the semigroup of values S_P. Let $\alpha, \beta \in S_P$ be, respectively, the smallest positive element and the conductor generator. We have $\sum \beta_i = 5$ and $3 \leqslant \sum \alpha_i = \dim \widetilde{\mathcal{O}}_P/\mathfrak{m}_P\widetilde{\mathcal{O}}_P < \dim \widetilde{\mathcal{O}}_P/\mathfrak{m}_P = 4$ where the first inequality comes from the fact that P is non-Gorenstein and the other from $\mathfrak{m}_P \neq \mathfrak{c}_P$. It follows that $\sum \alpha_i = 3$ and P has at most three branches. If it has three, we have $\alpha = (1, 1, 1)$ and hence $\beta = (n, n, n)$ for some $n \in \mathbb{N}^*$ which cannot happen.

If it has two branches, we have $\alpha = (1, 2)$ and three possibilities for β, that is, $(1, 4)$, $(2, 3)$, and $(3, 2)$. Since $(0, 3)$ and $(2, 1)$ cannot belong to S_P, we can discard first and third possibilities using [5, Lemma 4.1.1] and therefore $S_P = \{(0, 0), (1, 2)\} \cup \{(2, 3) + \mathbb{N}^2\}$. So let P_1 and P_2 be the two points of $\mathbb{P}^1$ over P. Consider the divisor $\mathfrak{a} := \mathcal{O}_C(1, x)$ on C where x is the identity function at finite points and $A_P := \{(v_{P_1}(z), v_{P_2}(z)) \mid z \in \mathfrak{a}_P\}$. Supposing, without loss of generality, that the two branches are at finite distance, we have that $(0, 0), (1, 0), (1, 2), (2, 2), (2, 3)$ is a saturated sequence in A_P and since $(0, 0), (1, 2), (2, 3)$ is also saturated in S_P, it follows that $\deg_P \mathfrak{a} = 2$ (cf. [4, Proposition 2.11(iii)]). But $\mathfrak{a}$ has degree 1 at infinity and 0 elsewhere but P, and thus $\deg \mathfrak{a} = 3$.

For the unibranch case we have $S_P = \{0, 3, 5, \to\}$ and we can suppose P under the origin of $\mathbb{P}^1$ to conclude immediately that the same divisor $\mathfrak{a}$ on C has degree 2 at P, 1 at the infinity and 0 elsewhere, that is, has degree 3 and C is trigonal. □

The simplest examples of curves of genus 3 satisfying cases 1 and 4 are, respectively, the parametrized rational curves (x^3, x^4, x^5, x^7) and (x^3, x^5, x^6, x^7) both living in $\mathbb{P}^4$, the first of which with two singular points and the second one with a unique.

3.2. The zero Maroni case

From now on, trigonal curves will always have genus larger or equal to 4. We begin by introducing the *Maroni invariant* within a theorem statement, which is proved in [12, Theorem 2.1 and Lemma 2.2]. We recall, for it, that linear systems which compute gonality are always spanned, and if the gonality is less than or equal to 3, they are also complete.

Theorem 3.2. *Let $\mathcal{L} = |\mathfrak{a}|$ be a g_3^1 on a trigonal curve C of genus g and $x \in H^0(C, \mathfrak{a}) \setminus k$. Then there exist integers $n \geqslant m \geqslant 0$ with $m + n = g - 2$ such that*

$$H^0(C, \omega) = \langle 1, x, \ldots, x^n, y, xy, \ldots, x^m y \rangle$$

for some canonical divisor ω and $y \in H^0(C, \omega)$. The constant m only depends on the linear system and is called the Maroni invariant *of the g_3^1.*

A geometric look on the above result lead us to know the ambient where lies the canonical model of C, if on a cone ($m = 0$), or on a scroll (m positive). Nothing prevents, at least in principle, that it can live on both surfaces or even on distinct scrolls, since m is an invariant of the g_3^1 and not of the curve. If the curve is Gorenstein, this does not happen (cf. [12,17]) but the whole non-Gorenstein problem escapes from our aim, for we would have to exhaust simultaneously zero and positive cases, which we did only for the first one. Anyway, we can define the *Maroni invariant of a trigonal curve* to be the smallest one among all of the g_3^1's of the curve, and the title of this work will finally make sense.

As promised we will study a zero Maroni trigonal curve C by means of its canonical model C'. For this, we will express the degree of C' in terms of the sum of abysses—which will be the parameter we will analyze—slightly modifying [8]'s formula by local duality.

Lemma 3.3. $\deg(C') = 2g - 2 - \eta$.

Proof. Since C and C' are birationally equivalent, we have $\deg(C') = \deg_{\widetilde{C}}(\omega \cdot \widetilde{\mathcal{O}})$. From [8, Lemma 2, p. 534] holds the first equality below, which we develop in the sequence:

$$
\begin{aligned}
\deg_{\widetilde{C}}(\omega \cdot \widetilde{\mathcal{O}}) &= (g-1) + \left(g - \sum_{P \in C_{\mathrm{sing}}} \delta_P\right) + \left(\left(\sum_{P \in C_{\mathrm{sing}}} \dim \mathcal{O}_P/\mathfrak{c}_P\right) - 1\right) \\
&= 2g - 2 - \left(\sum_{P \in C_{\mathrm{sing}}} \dim \widetilde{\mathcal{O}}_P/\mathcal{O}_P - \dim \mathcal{O}_P/\mathfrak{c}_P\right) \\
&= 2g - 2 - \left(\sum_{P \in C_{\mathrm{sing}}} \dim \widetilde{\mathcal{O}}_P/\mathcal{O}_P - \dim \widetilde{\mathcal{O}}_P/\omega_P\right) \\
&= 2g - 2 - \left(\sum_{P \in C_{\tilde{n}.\mathrm{gor}}} \dim \omega_P/\mathcal{O}_P\right) \\
&= 2g - 2 - \eta
\end{aligned}
$$

and we are done. □

We will define a point P of a curve to be *almost Gorenstein* provided that $\mathfrak{m}_P \omega_P = \mathfrak{m}_P$ for any P-normalized canonical divisor ω, or equivalently (cf. [6, Definition–Proposition 20 and Proposition 28]), P is Gorenstein or $\mu_P := \dim \mathcal{O}'_P/\omega_P = 1$ and we naturally extend the definition to a curve, which will be called *almost Gorenstein* if their points are so. As it turns out, mixing intrinsic and extrinsic degree formulas, we can conclude with the following theorem.

Theorem 3.4. *Every trigonal curve with zero Maroni invariant is almost Gorenstein with at most one non-Gorenstein point.*

Proof. Let C be a curve that fulfills the hypothesis and C' its canonical model. From Theorem 3.2, C' lies on a cone $S \subset \mathbb{P}^{g-1}$ where g is the genus of C. From Lemma 3.3 and formula (3), we can express the genus g' of the canonical model by the equation

$$g' = (q-1)(d'-1) - \frac{1}{2} q(q-1)(g-2) \tag{4}$$

with $d' = 2g - 2 - \eta$ and q being the smallest integer larger or equal to $d'/(g-2)$. But if C is non-Gorenstein, we have $g \leqslant d' \leqslant 2g - 3$ (since it is not basic hyperelliptic) which gives the possibilities $q = 2$ or 3. If $q = 2$ we have $g' = (d'-1) - (g-2) = g - 1 - \eta$ implying

$g - g' = 1 + \eta$ and it follows from Theorem 1.4(ii) that C admits only one non-Gorenstein point, which also satisfies $\mu_P = 1$. But the case $q = 3$ only happens if $d' = 2g - 3$, which lead us to the equality $\eta = 1$, i.e., a unique non-Gorenstein point P, and taking $d' = 2g - 3$ in (4) we have $g - g' = 2$ and then $\mu_P = 1$ and C is almost Gorenstein. $\square$

The reader can realize that the only fact we used on the above proof was the inclusion of the canonical model on a cone since we have not dealt with the g_3^1 of the curve. In other words, canonical models of curves with more than one non-Gorenstein point do not fit on a cone.

This must be understood. It is clear that there are of course a lot of curves lying on a cone, being canonical blowups of curves with so many non-Gorenstein points as desired: it suffices to apply replacing methods starting from the curve we want to be the blowup (we will do it later). But the curve on the cone will only be isomorphic to the canonical model of the constructed non-Gorenstein curve, which will certainly live in another space and never on a cone.

Proposition 3.5. *Let C be a trigonal non-Gorenstein curve carrying a zero Maroni g_3^1 given by $|\mathcal{O}_C(1, x)|$. Then, if the genus of C' is positive, the induced linear system $\mathcal{L}' := |\mathcal{O}_{C'}(1, x)|$ is a g_2^1 of C', necessarily with base point if $\eta \geqslant 3$. Otherwise, i.e., $C' \cong \mathbb{P}^1$, $\mathcal{L}'$ is a g_1^1.*

Proof. Let g and g' be, respectively, the genus of C and C'. We have that C' is a curve of degree $2g - 2 - \eta$ lying on the cone $S \subset \mathbb{P}^{g-1}$. From the previous theorem's proof, we have $g = g' + \eta + 1$, and hence C' is a curve of degree $2g' + \eta$ on the cone $\mathbb{P}^{g'+\eta}$. If $\sum_{i=0}^{l} c_i(x) y^i$ is the plane equation of C' on the cone, we have already seen that $\deg c_i(x) \leqslant 2g - 2 - \eta - i(g - 2)$ and we have $l \leqslant 2$ since $g \geqslant 4$ and $\eta \geqslant 1$. If $\eta = 1$ then C' is a curve of degree $2g' + 1$ on the cone S of $\mathbb{P}^{g'+1}$ and is either basic hyperelliptic if the vertex is singular (cf. Theorem 2.1(iii)) and we thus have $|\mathcal{O}_{C'}(1, x)|$ as the unique g_2^1 of C', or, otherwise, is given by an equation of the form $c_2 y^2 + c_1(x) y + c_0(x) = 0$ with $c_2 \neq 0$ as says the proof of implication (iii) $\Rightarrow$ (ii) of the mentioned theorem. In this case, the lines of the cone cut out a g_3^1 with base point at the vertex, which is a nonsingular point of C' and hence removable, what implies that $|\mathcal{O}_{C'}(1, x)|$ is the g_2^1 obtained with the removal of the base point. If $\eta = 2$ then C' is a curve of degree $2g' + 2$ in $\mathbb{P}^{g'+2}$ and the result follows from Theorem 2.4, where we must include the case $g' = 1$ in the application of the theorem. And if $\eta \geqslant 3$, we have $c_2(x) = 0$ and the plane equation for C' is reduced to $c_1(x) y + c_0(x) = 0$, with $\deg c_1(x) \leqslant g' + 1$ and $\deg c_0(x) \leqslant 2g' + \eta$ and we repeat the same argument of (iii) $\Rightarrow$ (ii) of Theorem 2.1 to see that, if $g' \geqslant 1$, then C' is rational with a unique singular point, whose maximal ideal is the conductor, and $|\mathcal{O}_{C'}(1, x)|$ is exactly the unique base point g_2^1 of C', verifying that the arguments extend to the case $g' = 1$, as can be checked. And, finally, if $\eta = g - 1$, i.e., $C' \cong \mathbb{P}^1$, then $\deg c_1(x) = 1$ and the result follows. $\square$

Corollary 3.6. *If the genus of the canonical model of a trigonal curve with zero Maroni invariant is greater or equal to 2, then the curve admits a unique Maroni zero g_3^1. If the canonical model is basic hyperelliptic, then the g_3^1 is unique at all.*

Proof. Follows from the uniqueness of the g_2^1 of C' if $g' \geqslant 2$ (cf. Corollary 2.3 and [15, Theorem 2.1]) and from the fact that if two g_3^1's in C induce, in the way of proposition, the same g_2^1 in the canonical model, they coincide; if C' is basic hyperelliptic, then it is non-Gorenstein and cannot live on a scroll, that is, every g_3^1 of C has zero Maroni invariant. □

One says a point P of a curve is *Kunz* if $\eta_P = 1$. We will extend the concept to a curve, which will be said *Kunz* if the same holds for all of its non-Gorenstein points. Then Gorenstein curves will be vacuously Kunz. With this extension, we can have a converse of what has been said, which also generalizes [12, Theorem 3.5] the following theorem.

Theorem 3.7. *Let C be a Kunz curve of genus $g \geqslant 5$. Then C is trigonal with zero Maroni invariant if and only if C' lies on a cone $S \subset \mathbb{P}^{g-1}$.*

Proof. The converse when C is non-Gorenstein, is our only remaining task. So let us suppose $C' \subset S \subset \mathbb{P}^{g-1}$. Then, as we have mentioned, C has a unique non-Gorenstein point, say, P. Since C is Kunz, we have $\eta = \eta_P = 1$. Then setting $d' = 2g - 3$ and $q = 3$ in Eq. (4), as has already been done, we conclude that $g - g' = 2$. Then C' is a curve of genus $2g' + 1$ on the cone $S \subset \mathbb{P}^{g'+1}$ and has gonality 2 from the proof of preceding proposition.

Let $|\mathfrak{a}'|$ where $\mathfrak{a}' := \mathcal{O}_{C'}(1, x)$ for some $x \in k(C')$ be the g_2^1 of C'; set $\mathfrak{a} := \mathcal{O}_C(1, x) \in \mathrm{Div}(C)$ and let $\psi : C' \to C$ be the morphism of Theorem 1.3. Since it is birational, ψ preserves cohomology by direct images and, in particular, we have that $H^0(C, \psi_*(\mathfrak{a}')) = \langle 1, x\rangle$, and thus $\mathcal{O} < \mathfrak{a} \leqslant \psi_*(\mathfrak{a}')$, and $\deg_C \psi_*(\mathfrak{a}') = \deg_{C'} \mathfrak{a}' + g - g' = 4$.

Since $h^0(C, \mathfrak{a}) = 2$, it follows that $\deg_C \mathfrak{a}$ cannot be 2 for C is neither hyperelliptic (it is non-Gorenstein!) nor basic hyperelliptic since $C' \ncong \mathbb{P}^1$ (cf. Corollary 2.3). Let us show that $\mathfrak{a} \neq \psi_*(\mathfrak{a}')$ and the degree will be 3.

Suppose $\mathfrak{a} = \psi_*(\mathfrak{a}')$, then we would have $\mathcal{O}'_P \subset \mathfrak{a}_P$ because $\mathcal{O}'_P = \bigcap_{P'|P} \mathcal{O}_{C',P'} = \psi_*(\mathcal{O}_{C'})_P \subset \psi_*(\mathfrak{a}')_P$. But $\mathfrak{a}_P = \mathcal{O}_P + x\mathcal{O}_P$ and we can choose x such that $x \notin \widetilde{\mathcal{O}}_P$ taking $(x - x(\widetilde{P}))^{-1}$ with $\widetilde{P}|P$ if necessary, forcing the inclusion $\mathcal{O}'_P \subset \mathcal{O}_P$, i.e., P would be Gorenstein.

It follows that $|\mathfrak{a}|$ is a g_3^1 of C, which is thereby trigonal. If C' is basic hyperelliptic, we are done. Otherwise, if C' is hyperelliptic, since $g' \geqslant 3$, we have that the g_2^1 is unique, and by the proof of last proposition, uniquely determined by the rule of the cone, i.e., by the canonical linear system of C. Since the g_3^1 of C was induced by the g_2^1 of C', it follows that its Maroni invariant equals zero. □

So far all Kunz curves with which we have been dealing here are almost Gorenstein. In fact, this is true in general (cf. [6, Proposition 21]) and so $\eta_P = 1$ implies $\mu_P = 1$, as we have pointed out in the beginning of this section. Besides, the reader must realize that the hypothesis on the genus of the curve was not needed in the above proof. We only did it in order to generalize the known result.

A simple example of a trigonal curve with zero Maroni invariant is the monomial curve $C = (x^5, x^6, x^7, x^9, x^{10}) \subset \mathbb{P}^5$ with genus $g = 5$, whose canonical model is $C' = (x^4, x^5, x^6, x^7) \subset \mathbb{P}^4$ of genus $g' = 3$. We see that $|\mathcal{O}_{C'}(1, x)|$ is a g_2^1 of C' with base point at the origin of $\mathbb{P}^4$, and thus C' is basic hyperelliptic, and $\mathcal{L} = |\mathcal{O}(1, x)|$ is a g_3^1 of C with base point of generic degree 2 at the origin of $\mathbb{P}^5$. As we have seen, the simple fact that C' is non-Gorenstein is enough to conclude that $\mathcal{L}$ has zero Maroni invariant, but we can also deduce it from [12, Lemma 2.3] observing that $\mathcal{O}(1, x) \cap \mathcal{O}(1, x^{-1})$ has degree 1 at the origin since both add the power x^8 to the local ring.

The curve C lies on a surface of $\mathbb{P}^5$ given by the equations

$$\operatorname{rank}\begin{pmatrix} X_1X_2X_4 \\ X_2X_3X_5 \end{pmatrix} < 2 \quad \text{and} \quad {X_1}^2 = X_0X_5,$$

which, as must be, is singular at the origin (the unique non-Gorenstein point of C) because at this point the Jacobian matrix has rank $1 \neq 3 = 5 - 2$.

It is important to observe that C is a curve of degree $2g$ in $\mathbb{P}^g$ and in this sense behaves as a canonical curve and only fails to be so because it has genus 5 and not 6 (cf. [10, Corollary, p. 189]).

And the example motivates the next result. For it, we will say that a curve is *unibranch* if the same holds for all of its points.

Theorem 3.8. *A unibranch curve C of genus g is trigonal with basic hyperelliptic canonical model if and only if there exists a point $P \in C$ with semigroup of values $S_P = \{0, g, g+1, \ldots, 2g-\eta-2, 2g-\eta, \rightarrow\}$ for some integer η such that $1 \leqslant \eta \leqslant g-3$.*

Proof. $\Rightarrow$. Let P be the non-Gorenstein point of C and let $\eta = \eta_P$ be the abyss. We have

$$\begin{aligned} \dim \widetilde{\mathcal{O}}_P/\mathfrak{c}_P &= \dim \widetilde{\mathcal{O}}_P/\omega_P + \dim \omega_P/\mathcal{O}_P + \dim \mathcal{O}_P/\mathfrak{c}_P \\ &= 2\dim \widetilde{\mathcal{O}}_P/\omega_P + \dim \omega_P/\mathcal{O}_P = 2(g-\eta) + \eta = 2g - \eta \end{aligned}$$

and the order of the conductor is $2g - \eta$. If P' is the unique non-Gorenstein point of C' then we have necessarily $P'|P$ since otherwise C would have at least two non-Gorenstein points, which cannot happen because of Theorem 3.4. But $\mathfrak{c}_{P'} = \mathfrak{m}_{P'}$ and hence $S_{P'} = \{0, g'+1, \rightarrow\} = \{0, g-\eta, \rightarrow\}$. Therefore, the integers from 1 to $g-\eta-1$ cannot belong to $v_{\widetilde{P}}(\omega_P)$, where $\widetilde{P}$ is the nonsingular point of $\widetilde{C}$ over P, and thus the integers from g to $2g-\eta-2$ must belong to S_P. Since $\dim \widetilde{\mathcal{O}}_P/\mathcal{O}_P = g$, the implication follows.

$\Leftarrow$. Since g is the genus of C and $1 \leqslant \eta \leqslant g-3$, we have that C' is basic hyperelliptic. To see that it is also trigonal, we reapply the same argument used at the end of the proof of Theorem 3.1. □

3.3. *Base points*

In this last subsection we will speak within a broader context, dealing with trigonal curves regardless of its Maroni invariant.

A question that makes sense in non-Gorenstein case is whether a trigonal curve is able to carry a g_3^1 with more than one base point. This is not possible if the curve is Gorenstein, for the g_3^1 is cut out by lines. In the general case we begin with the following examples.

Let C be a curve obtained from $\mathbb{P}^1$, which will be its canonical model, replacing the origin and the infinity, respectively, by the non-Gorenstein points P_0 and P_∞ both with singularity degree 2 , i.e., $\mathcal{O}_{P_0} = k \oplus \mathfrak{m}_0{}^3$ and $\mathcal{O}_{P_\infty} = k \oplus \mathfrak{m}_\infty{}^3$. Let x be the rational function identity at finite distance. It is easy to see that $|\mathcal{O}_C(1,x)|$ is a g_3^1 of C (which is hence trigonal) with base points P_0 and P_∞ since the stalks of $\mathcal{O}_C(1,x)$ at both points are not principal. The curve C also admits another g_3^1, which is base point free, given by $|\mathcal{O}_C(1,x^3)|$.

If, on the other hand, we replace 0 and ∞, respectively, by the singular points whose local rings are $k \oplus \mathfrak{m}_0{}^4$ and $k \oplus \mathfrak{m}_\infty{}^2$ we still have a trigonal curve carrying a g_3^1 with two base points, one of them Gorenstein, i.e., the one who lies under the infinity. Actually, this is essentially the same example of one of the two curves we offered at the end of Section 3.1, though with genus 4. We would have $C = (x^4, x^5, x^6, x^7, x^9)$ and C' singular elliptic. These examples motivate the hypothesis we make below.

Proposition 3.9. *Every base point of a g_3^1 of a trigonal non-Gorenstein curve, whose canonical model has degree greater or equal to* 2*, is necessarily non-Gorenstein. In this case, for a g_3^1 of a trigonal non-Gorenstein curve C to admit two base points, it is necessary that it induces in C' a pencil without any total ramification.*

Proof. If $\mathcal{L} = |\mathcal{O}_C(1,x)|$ is a g_3^1 of a non-Gorenstein trigonal curve C and P a Gorenstein point of the curve, which is also a base point of $\mathcal{L}$ then P is also a base point of $\mathcal{L}' = |\mathcal{O}_{C'}(1,x)|$. But from Proposition 3.5 for the zero Maroni case and the fact that $\mathcal{L}'$ is base point free if C' lies on a scroll (cf. [17] for details), since $g' \geqslant 2$, we would have C' basic hyperelliptic and P non-Gorenstein as the only possible choice, and follows the first statement. For the second, let us consider $P' \in C'$ a totally ramified point of $\mathcal{L}'$. For $\mathcal{L}$ to admit two (non-Gorenstein) base points, we have that P' needs to be over a (possibly third) non-Gorenstein point, say, P (otherwise, $\mathcal{L}$ would have degree at least 4). Consider then the curve C_1 such that P is the unique non-Gorenstein point of C_1 and which also has C' as canonical model. Since $g' \geqslant 2$, the linear system $\mathcal{L}_1 := |\mathcal{O}_{C_1}(1,x)|$ cannot be a g_2^1 of C_1—otherwise, it would be basic hyperelliptic, but $C_1' = C' \ncong \mathbb{P}^1$—and thus is a pencil of degree greater or equal to 3. But since $P'|P$, we have P totally ramified with respect to $\mathcal{L}_1$, which forbids $\mathcal{L}$ to have another base point. □

Theorem 3.4, Proposition 3.5, Corollary 3.6, and Proposition 3.9 together prove the theorem announced in the introduction. The same questions can certainly be answered in the same way for the positive Maroni case, but we leave them for a forthcoming work.

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References

[1] E. Ballico, Maximal degree subsheaves of torsion free sheaves on singular projective curves, Trans. Amer. Math. Soc. 353 (9) (2001) 3617–3627.

[2] E. Ballico, On the reflexivity of canonical models of non-Gorenstein curves, Internat. Math. J. 1 (4) (2002) 363–365.

[3] E. Ballico, Trigonal Gorenstein curves and Weierstrass points, Tsukuba J. Math. 26 (2002) 133–144.

[4] V. Barucci, M. D'Anna, R. Fröberg, Analytically unramified one-dimensional semilocal rings and their value semigroups, J. Pure Appl. Algebra 147 (2000) 215–254.

[5] V. Barucci, M. D'Anna, R. Fröberg, The semigroup of values of a one-dimensional local ring with two minimal primes, Comm. Algebra 28 (8) (2000) 3607–3633.

[6] V. Barucci, R. Fröberg, One-dimensional almost Gorenstein rings, J. Algebra 188 (1997) 418–442.

[7] A. Grothendieck, J. Dieudonné, Élements de Géométrie Algebrique I, in: Grundlehren Math. Wiss., vol. 166, Springer-Verlag, Heidelberg, 1971.

[8] D. Eisenbud, J. Koh, M. Stillmann (appendix with J. Harris), Determinantal equations for curves of high degree, Amer. J. Math. 110 (1988) 513–539.

[9] A. Maroni, Le Serie Lineare Speciale Sulle Curve Trigonali, Ann. Mat. Pura Appl. 25 (1946) 333–341.

[10] M. Rosenlicht, Equivalence relations on algebraic curves, Ann. of Math. 56 (1952) 169–191.

[11] R. Rosa, Non-classical trigonal curves, J. Algebra 225 (2000) 359–380.

[12] R. Rosa, K.-O. Stöhr, Trigonal Gorenstein curves, J. Pure Appl. Algebra 174 (2002) 187–205.

[13] J.-P. Serre, Groupes Algebriques et Corps de Classes, Hermann, Paris, 1959.

[14] K.-O. Stöhr, On the poles of regular differentials of singular curves, Bol. Soc. Brasil. Mat. 24 (1993) 105–135.

[15] K.-O. Stöhr, Hyperelliptic Gorenstein curves, J. Pure Appl. Algebra 135 (1999) 93–105.

[16] K.-O. Stöhr, Local and global zeta-functions of singular algebraic curves, J. Number Theory 71 (1998) 172–202.

[17] K.-O. Stöhr, P. Viana, Weierstrass gap sequences and moduli varieties of trigonal curves, J. Pure Appl. Algebra 81 (1992) 63–82.